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Waveform Relaxation Methods for Functional Differential Systems of Neutral Type

Z. Jackiewicz*

M. Kwapisz

Institute of Mathematics, University of Gdansk, 80-952 Gdansk, Poland

and

E. Lo

Planetary Geology Group, Department of Geology, Arizona State University, Tempe, Arizona 85287-1804

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We investigate continuous-time and discretized waveform relaxation iterations for functional differential systems of neutral type. It is demonstrated that continuous-time iterations converge linearly for neutral equations and superlinearly when the right hand side is independent of the history of the derivative of the solution. The error bounds for discretized iterations are also obtained and some implementation aspects are discussed. Numerical results are presented which indicate a potential speedup of this technique as compared with the classical approach based on discrete variable methods. © 1997 Academic Press

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1. INTRODUCTION

Consider the initial-value problem for Volterra functional differential systems of neutral type

$$\begin{cases} y'(t) = f(t, y(\cdot), y'(\cdot)), & t \in I_a = [a, b], \\ y(t) = g(t), & t \in [\alpha, a], \end{cases} \tag{1.1}$$

$\alpha \le a < b$, where $f : I_\alpha \times C_g[I_\alpha, R^n] \times \tilde{C}_{g'}[I_\alpha, R^n] \to R^n$ and $g : [\alpha, a] \to R^n$ is a given initial function which is continuous with its first derivative. Here, $I_\alpha = [\alpha, b]$ and $C_g[I_\alpha, R^n]$ stands for the class of continuous functions defined on I_α with values in R^n which are equal to g for $t \in [\alpha, a]$. Similarly, $\tilde{C}_{g'}[I_\alpha, R^n]$ stands for the class of piecewise continuous functions defined on I_α with values in R^n which are equal to g' for $t \in [\alpha, a]$. We will assume that f has the following properties

(a) For given $y \in C_g[I_\alpha, R^n]$ and $z \in C_{g'}[I_\alpha, R^n]$ the mapping $t \to f(t, y(\cdot), z(\cdot))$ is continuous on I_α.

(b) There exist constants $\tilde{L} \ge 0$ and $0 \le \tilde{K} < 1$ such that

$$\| f(t, y(\cdot), z(\cdot)) - f(t, \bar{y}(\cdot), \bar{z}(\cdot)) \| \le \tilde{L} \, \|y - \bar{y}\|_t + \tilde{K} \, \|z - \bar{z}\|_t$$

for any $t \in I_\alpha$, $y, \bar{y} \in C_g[I_\alpha, R^n]$, and $z, \bar{z} \in C_{g'}[I_\alpha, R^n]$.

In the above condition $\|y\|_t$ stands for $\sup\{\|y(s)\| : s \in [\alpha, t]\}$, where $\|\cdot\|$ is some norm on R^n.

These assumptions guarantee the existence of a unique solution y of (1.1). This is a direct consequence of the Banach contraction mapping principle (compare also [16]).

In this paper we study the numerical solution of (1.1) by waveform relaxation (WR) techniques. Such techniques were first proposed by Lelarasmee [18] and Lelarasmee *et al.* [19] for time domain analysis of large differential systems modelling electrical networks. They were further studied for ordinary differential equations by Nevanlinna and his co-workers [24–28], Skeel [32], Lie and Skålin [20], Bellen and Zennaro [2], and Bellen, Jackiewicz, and Zennaro [5]. We also refer to a survey paper with emphasis on simulation of large electrical circuits by White *et al.* [35] and to the book by White and Sangiovanni-Vincentelli [34]. A related technique of time-point relaxation was studied by Lie and Skålin [20], Bellen, Jackiewicz, and Zennaro [3, 4], and Jackiewicz [12]. Bjørhus [6] used this technique for delay differential equations

$$\begin{cases} y'(t) = f(t, y(t), y(\theta(t))), & t \geq 0, \\ y(0) = y_0, \end{cases}$$

$0 \leq \theta(t) \leq t$, to uncouple the dependence between $y(t)$ and $y(\theta(t))$. He studied the convergence properties of an iteration scheme in which the delay term is read from the previous iteration.

Given a splitting function

$$F : I_\alpha \times C_g[I_\alpha, R^n] \times C_g[I_\alpha, R^n] \times C_{g'}[I_\alpha, R^n] \times C_{g'}[I_\alpha, R^n] \to R^n$$

such that F is continuous for $t \in I_a$,

$$F(t, y(\cdot), y(\cdot), z(\cdot), z(\cdot)) = f(t, y(\cdot), z(\cdot)),$$

for $t \in I_a$, $y \in C_g[I_\alpha, R^n]$, $z \in C_{g'}[I_\alpha, R^n]$, and

$$\| F(t, y_1(\cdot), y_2(\cdot), z_1(\cdot), z_2(\cdot)) - F(t, \bar{y}_1(\cdot), \bar{y}_2(\cdot), \bar{z}_1(\cdot), \bar{z}_2(\cdot)) \|$$
$$\leq L_1 \|y_1 - \bar{y}_1\|_t + L_2 \|y_2 - \bar{y}_2\|_t + K_1 \|z_1 - \bar{z}_2\|_t + K_2 \|z_2 - \bar{z}_2\|_t$$

$L_1, L_2, K_1, K_2 \geq 0$, $K_1 + K_2 < 1$, for $t \in I_a$, $y_1, \bar{y}_1, y_2, \bar{y}_2 \in C_g[I_\alpha, R^n]$ and $z_1, \bar{z}_1, z_2, \bar{z}_2 \in C_{g'}[I_\alpha, R^n]$. The general form of a continuous-time WR iteration is

$$z^{(\nu)}(t) = F(t, y^{(\nu)}(\cdot), y^{(\nu-1)}(\cdot), z^{(\nu)}(\cdot), z^{(\nu-1)}(\cdot)), \tag{1.2}$$

$t \in I_a$, $\nu = 1, 2, \ldots,$ with $z^{(\nu)}(t) = (d/dt)y^{(\nu)}(t)$, where $y^{(0)} \in C_g[I_\alpha, R^n]$ and $z^{(0)} \in \tilde{C}_{g'}[I_\alpha, R^n]$ are given initial approximations to y and y', respectively, such that $z^{(0)}(t) = (d/dt)y^{(0)}(t)$. Usually, $z^{(0)}(t) = 0$, $y^{(0)}(t) = g(a)$, for $t \in I_a$, or $z^{(0)}(t) = g'(a)$, $y^{(0)}(t) = g(a) + g'(a)(t-a)$, for $t \in I_a$. Examples of such iterations are WR Gauss–Jacobi iterations which correspond to the splitting function F defined by

$$\begin{aligned} &F_i(t, y(\cdot), \bar{y}(\cdot), z(\cdot), \bar{z}(\cdot)) \\ &\quad = f_i(t, \bar{y}_1(\cdot), \ldots, \bar{y}_{i-1}(\cdot), y_i(\cdot), \bar{y}_{i+1}(\cdot), \ldots, \bar{y}_n(\cdot), \\ &\qquad z_1(\cdot), \ldots, \bar{z}_{i-1}(\cdot), z_i(\cdot), \bar{z}_{i+1}(\cdot), \ldots, \bar{z}_n(\cdot)), \end{aligned}$$

$i = 1, 2, \ldots, n$, or WR Gauss–Seidel iterations which correspond to the splitting function F defined by

$$\begin{aligned} F_i(t, y(\cdot), \bar{y}(\cdot), z(\cdot), \bar{z}(\cdot)) = f_i(t, y_1(\cdot), \ldots, y_i(\cdot), \bar{y}_{i+1}(\cdot), \ldots, \bar{y}_n(\cdot), \\ z_1(\cdot), \ldots, z_i(\cdot), \bar{z}_{i+1}(\cdot), \ldots, \bar{z}(\cdot)), \end{aligned}$$

$i = 1, 2, \ldots, n$. Consider also the iterations

$$\tilde{z}^{(\nu)}(t) = F\big(t, \tilde{y}^{(\nu)}(\cdot), \tilde{y}_h^{(\nu-1)}(\cdot), \tilde{z}^{(\nu)}(\cdot), \tilde{z}_h^{(\nu-1)}(\cdot)\big), \tag{1.3}$$

$t \in I_a$, $\nu = 1, 2, \ldots,$ with $\tilde{z}^{(\nu)}(t) = (d/dt)\tilde{y}^{(\nu)}(t)$, where $\tilde{y}_h^{(0)} \in C_g[I_\alpha, R^n]$ is a given approximation to $\tilde{y}^{(0)} = y^{(0)}$, and $\tilde{z}_h^{(0)} \in C_{g'}[I_\alpha, R^n]$ is a given approximation to $\tilde{z}^{(0)} = z^{(0)}$. We denote by $\tilde{y}_h^{(\nu)}$ a continuous approximation to $\tilde{y}^{(\nu)}$ and by $\tilde{z}_h^{(\nu)}$ a piecewise continuous approximation to $\tilde{z}^{(\nu)}$. These approximations are obtained by application to the system (1.3) of some numerical techniques (for example, Runge–Kutta, linear multistep, or predictor-corrector methods) implemented on constant or variable stepsize meshes. We denote by h a discretization parameter which is usually equal to the stepsize for constant stepsize implementation or a minimum stepsize for variable stepsize implementation.

To study convergence properties of the iterations $\tilde{y}_h^{(\nu)}$ and $\tilde{z}_h^{(\nu)}$ we split the global errors $\|y - \tilde{y}_h^{(\nu)}\|_t$ and $\|z - \tilde{z}_h^{(\nu)}\|_t$, $z = y'$, into three components of the form

$$\left\| y - \tilde{y}_h^{(\nu)} \right\|_t \le \left\| y - y^{(\nu)} \right\|_t + \left\| y^{(\nu)} - \tilde{y}^{(\nu)} \right\|_t + \left\| \tilde{y}^{(\nu)} - \tilde{y}_h^{(\nu)} \right\|_t \tag{1.4}$$

and

$$\left\| z - \tilde{z}_h^{(\nu)} \right\|_t \le \left\| z - z^{(\nu)} \right\|_t + \left\| z^{(\nu)} - \tilde{z}^{(\nu)} \right\|_t + \left\| \tilde{z}^{(\nu)} - \tilde{z}_h^{(\nu)} \right\|_t. \tag{1.5}$$

The bounds for the errors $\|y - y^{(\nu)}\|_t$ and $\|z - z^{(\nu)}\|_t$ of the iterations $y^{(\nu)}$ and $z^{(\nu)}$ defined by (1.2) are obtained in Section 2. In turns out that, in general, $y^{(\nu)}$ and $z^{(\nu)}$ converge linearly to y and z, respectively, and that the convergence is superlinear if the function $F(t, y(\cdot), \bar{y}(\cdot), z(\cdot), \bar{z}(\cdot))$ is independent of $\bar{z}(\cdot)$ (i.e., $K_2 = 0$). The bounds for the errors $\|y^{(\nu)} - \tilde{y}^{(\nu)}\|_t$ and $\|z^{(\nu)} - \tilde{z}^{(\nu)}\|_t$ between the iterations $y^{(\nu)}$ and $z^{(\nu)}$ and the perturbed iterations $\tilde{y}^{(\nu)}$ and $\tilde{z}^{(\nu)}$ defined by (1.3) are studied in Section 3. These bounds depend on the global errors $\tilde{y}^{(\nu-1)} - \tilde{y}_h^{(\nu-1)}$ and $\tilde{z}^{(\nu-1)} - \tilde{z}_h^{(\nu-1)}$ of the numerical procedure used to compute approximations $\tilde{y}_h^{(\nu-1)}$ and $\tilde{z}_h^{(\nu-1)}$ to $\tilde{y}^{(\nu-1)}$ and $\tilde{z}^{(\nu-1)}$. The bounds for global errors $\|\tilde{y}^{(\nu)} - \tilde{y}_h^{(\nu)}\|_t$ and $\|\tilde{z}^{(\nu)} - \tilde{z}_h^{(\nu)}\|_t$ between the perturbed iterations $\tilde{y}^{(\nu)}$ and $\bar{z}^{(\nu)}$ and discretized iterations $\tilde{y}_h^{(\nu)}$ and $\bar{z}_h^{(\nu)}$ are studied in Section 4 for the general class of quasilinear multistep methods for (1.1). These methods include as special cases Runge–Kutta, linear multistep, and predictor-corrector methods for neutral equations (1.1) and were studied in [7–9] for constant stepsize implementation and in [10, 11, 13, 14, 21] for variable step/variable order implementation. In Section 5 we describe the algorithm to compute the discretized WR iterations $\tilde{y}_h^{(\nu)}$ and $\tilde{z}_h^{(\nu)}$ and discuss some implementation issues such as the choice of the window of integration and the choice of the tolerance used to compute $\tilde{y}_h^{(\nu)}$ and $\tilde{z}_h^{(\nu)}$. This tolerance will depend on the tolerance used to bound the local error of quasilinear multistep method and on the iteration index ν. Finally, in Section 6 we present and discuss the results of numerical experiments on the selection of test problems listed in [14] and describe plans for future research.

2. CONVERGENCE OF CONTINUOUS-TIME WR ITERATIONS $y^{(\nu)}$ AND $z^{(\nu)}$

Subtracting (1.2) from the equation

$$\begin{cases} z(t) = F(t, y(\cdot), y(\cdot), z(\cdot), z(\cdot)), & t \in Ia, \\ y(t) = g(t), \qquad z(t) = g'(t), & t \in [\alpha, a], \end{cases}$$

and using the Lipschitz condition for the function F we obtain

$$\begin{aligned} \|z - z^{(\nu)}\|_t \le L_1 \|y - y^{(\nu)}\|_t + L_2 \|y - y^{(\nu-1)}\|_t \\ + K_1 \|z - z^{(\nu)}\|_t + K_2 \|z - z^{(\nu-1)}\|_t, \end{aligned}$$

$t \in I_a$, $\nu = 1, 2, \dots$. Since

$$y(t) = \begin{cases} g(a) + \int_a^t z(s)\, ds, & t \in I_a, \\ g(t), & t \in [\alpha, a], \end{cases}$$

$$y^{(\nu)}(t) = \begin{cases} g(a) + \int_a^t z^{(\nu)}(s)\, ds, & t \in I_a, \\ g(t), & t \in [\alpha, a], \end{cases}$$

it follows that

$$\|y - y^{(\nu)}\|_t \le \int_a^t \|z - z^{(\nu)}\|_s\, ds,$$

and we get

$$\begin{aligned} v^{(\nu)}(t) \le L_1 \int_a^t v^{(\nu)}(s)\, ds + L_2 \int_a^t v^{(\nu-1)}(s)\, ds \\ + K_1 v^{(\nu)}(t) + K_2 v^{(\nu-1)}(t), \end{aligned}$$

$t \in I_a$, $\nu = 1, 2, \dots,$ where $v^{(\nu)}(t) := \|z - z^{(\nu)}\|_t$. Define the sequence

$u^{(\nu)}(t)$ by the equation

$$\begin{cases} u^{(\nu)}(t) = L_1 \int_a^t u^{(\nu)}(s)\,ds + L_2 \int_a^t u^{(\nu-1)}(s)\,ds \\ \qquad\qquad +K_1 u^{(\nu)}(t) + K_2 u^{(\nu-1)}(t), \\ u^{(0)}(t) = v^{(0)}(t), \end{cases} \tag{2.1}$$

$t \in I_a$, $\nu = 1, 2, \ldots$. It follows from the theory of integral inequalities that

$$v^{(\nu)}(t) \le u^{(\nu)}(t), \qquad t \in I_a, \quad \nu = 1, 2, \ldots . \tag{2.2}$$

Let

$$A = \frac{L_1}{1 - K_1}, \qquad B = \frac{L_2}{1 - K_1}, \qquad K = \frac{K_2}{1 - K_1}.$$

Then $K < 1$ and (2.1) can be written in the form

$$u^{(\nu)}(t) = A \int_a^t u^{(\nu)}(s)\,ds + B \int_a^t u^{(\nu-1)}(s)\,ds + K u^{(\nu-1)}(t).$$

Put

$$m(t) = \int_a^t u^{(\nu)}(s)\,ds,$$

$$h(t) = B \int_a^t u^{(\nu-1)}(s)\,ds + K u^{(\nu-1)}(t).$$

Then m satisfies the differential equation

$$\begin{cases} m'(t) = Am(t) + h(t), \quad t \in I_a \\ m(a) = 0, \end{cases}$$

whose solution is

$$m(t) = \int_a^t e^{A(t-s)} h(s)\,ds.$$

Since $u^{(\nu)}(t) = Am(t) + h(t)$ it follows that

$$u^{(\nu)}(t) = AB \int_a^t \int_a^s e^{A(t-s)} u^{(\nu-1)}(\xi)\,d\xi\,ds$$
$$+ AK \int_a^t e^{A(t-s)} u^{(\nu-1)}(s)\,ds + B \int_a^t u^{(\nu-1)}(s)\,ds + K u^{(\nu-1)}(t),$$

and changing the order of integration in the double integral appearing above we obtain

$$u^{(\nu)}(t) = M\int_a^t e^{A(t-s)}u^{(\nu-1)}(s)\,ds + Ku^{(\nu-1)}(t), \tag{2.3}$$

$t \in Ia$, $\nu = 1, 2, \dots,$ where $M = B + AK$. Multiplying (2.3) by $e^{-A(t-a)}$ and putting

$$\overline{u}^{(\nu)}(t) = e^{-A(t-a)}u^{(\nu)}(t)$$

we obtain

$$\begin{cases} \overline{u}^{(\nu)}(t) = M\int_a^t \overline{u}^{(\nu-1)}(s)\,ds + K\overline{u}^{(\nu-1)}(t), \\ \overline{u}^{(0)}(t) = e^{-A(t-a)}u^{(0)}(t), \end{cases} \tag{2.4}$$

$t \in I_a$, $\nu = 1, 2, \dots$. Define the linear operators P and Q by

$$(Pu)(t) = M\int_a^t u(s)\,ds, \qquad (Qu)(t) = Ku(t).$$

Then (2.4) can be rewritten in the form

$$\overline{u}^{(\nu)} = (P + Q)\overline{u}^{(\nu-1)},$$

$\nu = 1, 2, \dots,$ whose solution is

$$\overline{u}^{(\nu)} = (P + Q)^{\nu}\overline{u}^{(0)}.$$

Observe that $PQ = QP$. Hence,

$$(P + Q)^{\nu} = \sum_{i=0}^{\nu}\binom{\nu}{i}P^iQ^{\nu-i}.$$

It is easy to verify that

$$(P^iu)(t) = M^i\int_a^t \frac{(t-s)^{i-1}}{(i-1)!}\,u(s)\,ds.$$

Obviously, we have also $(Q^iu)(t) = K^iu(t)$ and it follows that

$$\overline{u}^{(\nu)}(t) = \sum_{i=0}^{\nu}\binom{\nu}{i}M^iK^{\nu-i}\int_a^t \frac{(t-s)^{i-1}}{(i-1)!}\,\overline{u}^{(0)}(s)\,ds, \tag{2.5}$$

$i = 1, 2, \ldots, P^0 = I$. Assume that $u^{(0)}(t) \leq Z_0$, $t \in I_a$. Then $\bar{u}^{(0)}(t) = e^{-A(t-a)} u^{(0)}(t) \leq Z_0$, $t \in I_a$, and it follows from (2.5) that

$$\bar{u}^{(\nu)}(t) \leq Z_0 \sigma^{(\nu)}(t), \tag{2.6}$$

$t \in I_a$, $\nu = 0, 1, \ldots,$ where $\sigma^{(\nu)}(t)$ is defined by

$$\sigma^{(\nu)}(t) = \sum_{i=0}^{\nu} \binom{\nu}{i} \frac{(M(t-a))^i}{i!} K^{\nu-i}. \tag{2.7}$$

Inequalities (2.6) and (2.2) yield

$$\|z - z^{(\nu)}\|_t \leq Z_0 e^{A(t-a)} \sigma^{(\nu)}(t).$$

We have also

$$\begin{aligned} \|y - y^{(\nu)}\|_t &\leq \int_a^t \|z - z^{(\nu)}\|_s \, ds \\ &\leq (b-a)\|z - z^{(\nu)}\|_t \leq Y_0 e^{A(t-a)} \sigma^{(\nu)}(t), \end{aligned}$$

$t \in I_a$, where $Y_0 = (b-a)Z_0$ is a bound on $\|y - y^{(0)}\|_t$. We summarize the above discussion in the following theorem.

THEOREM 1. *Assume that* $\|z - z^{(0)}\|_t \leq Z_0$, $t \in I_a$, *and let* $y^{(\nu)}$ *and* $z^{(\nu)}$ *be defined by* (1.2). *Then*

$$\|y - y^{(\nu)}\|_t \leq e^{A(t-a)} \sigma^{(\nu)}(t) Y_0,$$

and

$$\|z - z^{(\nu)}\|_t \leq e^{A(t-a)} \sigma^{(\nu)}(t) Z_0,$$

$t \in I_a$, $\nu = 0, 1, \ldots,$ *where* $Y_0 = (b-a)Z_0$, *and* $\sigma^{(\nu)}(t)$ *is defined by* (2.7).

Consider now the special case of (1.1) where the function f is independent of the history of the derivative of the solution $y'(\cdot)$. Then $K = 0$ and the bounds for $\|y - y^{(\nu)}\|_t$ and $\|z - z^{(\nu)}\|_t$ take the form

$$\|y - y^{(\nu)}\|_t \leq \exp(L_1(t-a)) \frac{(L_2(t-a))^\nu}{\nu!} Y_0,$$

$$\|z - z^{(\nu)}\|_t \leq \exp(L_1(t-a)) \frac{(L_2(t-a))^\nu}{\nu!} Z_0,$$

$t \in I_a$, $\nu = 0, 1, \ldots$. These bounds have the same form as the bounds obtained in the case of ordinary differential equations (compare, for example, [2]) and they show that the convergence of $y^{(\nu)}$ and $z^{(\nu)}$ to y and y' is superlinear. In the neutral case, however, where f depends on $y'(\cdot)$ (i.e., $0 < K < 1$) the convergence is only linear and the rate of convergence is K. We have the following theorem.

THEOREM 2. *Assume that* $0 < K < 1$. *Then*

$$\lim_{\nu \to \infty} \frac{\sigma^{(\nu+1)}(t)}{\sigma^{(\nu)}(t)} = K,$$

for $t \in I_a$ *and* $M \geq 0$.

Proof. The proof presented below was suggested to us by Colm O'Cinneide from Purdue University.

Define

$$\sigma_\mu^{(\nu)}(t) = \sum_{i=\mu}^{\nu} \frac{P(\nu, i)}{i!},$$

$$\tilde{\sigma}_\mu^{(\nu)}(t) = \sum_{i=\mu}^{\nu} \frac{P(\nu, i)}{(i+1)!},$$

$\mu = 0, 1, \dots,$ where

$$P(\nu, i) = \binom{\nu}{i} (M(t-a))^i K^{\nu - i}.$$

Observe that $\sigma_0^{(\nu)}(t) = \sigma^{(\nu)}(t)$. Routine calculations yield

$$\frac{\sigma^{(\nu+1)}(t)}{\sigma^{(\nu)}(t)} = K + M(t-a) \frac{\tilde{\sigma}_0^{(\nu)}(t)}{\sigma_0^{(\nu)}(t)},$$

and we will prove that

$$\lim_{\nu \to \infty} \frac{\tilde{\sigma}_0^{(\nu)}(t)}{\sigma_0^{(\nu)}(t)} = 0.$$

It can be verified that $P(\nu, i)/(i+1)!$ and $P(\nu, i)/i!$ are increasing for $i < \bar{\nu}$ and $i < \underline{\nu}$, respectively, where

$$\bar{\nu} = \left\lfloor \frac{-3 - Q + \sqrt{Q^2 + 6Q + 1 + 4Q\nu}}{2} \right\rfloor \approx \sqrt{Q\nu}$$

and

$$\underline{\nu} = \left\lfloor \frac{-2 - Q + \sqrt{Q^2 + 4Q + 4Q\nu}}{2} \right\rfloor \approx \sqrt{Q\nu}$$

for large ν, with $Q = Q(t) = M(t-a)/K$. Here $\lfloor S \rfloor$ stands for the integer part of S. This gives

$$\sum_{i=0}^{\mu-1} \frac{P(\nu,i)}{(i+1)!} \le \frac{\mu-1}{\bar{\nu}} \sum_{i=0}^{\bar{\nu}} \frac{P(\nu,i)}{(i+1)!} \le \frac{\mu-1}{\bar{\nu}} \tilde{\sigma}_0^{(\nu)}(t),$$

$$\sum_{i=0}^{\mu-1} \frac{P(\nu,i)}{i!} \le \frac{\mu-1}{\underline{\nu}} \sum_{i=0}^{\underline{\nu}} \frac{P(\nu,i)}{i!} \le \frac{\mu-1}{\underline{\nu}} \sigma_0^{(\nu)}(t),$$

and it follows that

$$\limsup_{\nu\to\infty} \left(\sum_{i=0}^{\mu-1} \frac{P(\nu,i)}{(i+1)!} \bigg/ \tilde{\sigma}_0^{(\nu)}(t) \right) = \limsup_{\nu\to\infty} \left(\sum_{i=0}^{\mu-1} \frac{P(\nu,i)}{i!} \bigg/ \sigma^{(\nu)}(t) \right) = 0$$

for every fixed integer μ, $\mu \ge 0$. These relations and

$$\frac{\tilde{\sigma}_\mu^{(\nu)}(t)}{\sigma_\mu^{(\nu)}(t)} = \frac{\tilde{\sigma}_0^{(\nu)}(t)\left(1 - \sum_{i=0}^{\mu-1}(P(\nu,i)/(i+1)!)/\overline{\sigma}_0^{(\nu)}(t)\right)}{\sigma_0^{(\nu)}(t)\left(1 - \sum_{i=0}^{\mu-1}(P(\nu,i)/i!)/\sigma_0^{(\nu)}(t)\right)}$$

imply that

$$\limsup_{\nu\to\infty} \frac{\tilde{\sigma}_\mu^{(\nu)}(t)}{\sigma_\mu^{(\nu)}(t)} = \limsup_{\nu\to\infty} \frac{\tilde{\sigma}_0^{(\nu)}(t)}{\sigma_0^{(\nu)}(t)}$$

for every fixed integer $\mu \ge 0$. Observe next that

$$\frac{P(\nu,i)}{(i+1)!} \le \frac{1}{\mu} \frac{P(\nu,i)}{i!}$$

for $i \ge \mu$. Summing both sides of these inequalities from μ to ν we get

$$\tilde{\sigma}_\mu^{(\nu)}(t) \le \frac{1}{\mu} \sigma_\mu^{(\nu)}(t)$$

and it follows that

$$\limsup_{\nu\to\infty} \frac{\tilde{\sigma}_0^{(\nu)}(t)}{\sigma_0^{(\nu)}(t)} = \limsup_{\nu\to\infty} \frac{\tilde{\sigma}_\mu^{(\nu)}(t)}{\sigma_\mu^{(\nu)}(t)} \le \frac{1}{\mu}$$

for every fixed integer $\mu \geq 0$. Passing with μ to infinity we obtain

$$\lim_{\nu\to\infty} \frac{\tilde{\sigma}_0^{(\nu)}(t)}{\sigma_0^{(\nu)}(t)} = 0,$$

and the theorem follows. ∎

3. CONVERGENCE OF PERTURBED WR ITERATIONS $\tilde{y}^{(\nu)}$ AND $\tilde{z}^{(\nu)}$

In this section we will establish the bounds for $\|y^{(\nu)} - \tilde{y}^{(\nu)}\|_t$ and $\|z^{(\nu)} - \tilde{z}^{(\nu)}\|_t$ which, when combined with the bounds for $\|y - y^{(\nu)}\|_t$ and $\|z - z^{(\nu)}\|_t$ obtained in Section 2, prove under certain conditions the convergence of $\tilde{y}^{(\nu)}$ and $\tilde{z}^{(\nu)}$ to y and y', respectively, as $\nu \to \infty$ and $h \to 0$.

Assume that the approximations $\tilde{y}_h^{(\nu-1)}$ and $\tilde{z}_h^{(\nu-1)}$ to $\tilde{y}^{(\nu-1)}$ and $\tilde{z}^{(\nu-1)}$ are already computed on the whole interval I_a and denote by $\epsilon_h^{(\nu-1)}$ and $e_h^{(\nu-1)}$ global errors of these approximations, i.e.,

$$\epsilon_h^{(\nu-1)}(t) = \tilde{y}^{(\nu-1)}(t) - \tilde{y}_h^{(\nu-1)}(t), \qquad e_h^{(\nu-1)}(t) = \tilde{z}_h^{(\nu-1)}(t) - \tilde{z}_h^{(\nu-1)}(t).$$

Subtracting (1.3) from (1.2) and using the Lipschitz condition on the function F we obtain

$$\begin{aligned}\|z^{(\nu)} - \tilde{z}^{(\nu)}\|_t \leq{}& L_1 \left\|y^{(\nu)} - \tilde{y}^{(\nu)}\right\|_t + L_2 \left\|y^{(\nu-1)} - \tilde{y}^{(\nu-1)}\right\|_t \\ &+ K_1 \left\|z^{(\nu)} - \tilde{z}^{(\nu)}\right\|_t + K_2 \left\|z^{(\nu-1)} - \tilde{z}^{(\nu-1)}\right\|_t \\ &+ L_2 \left\|\epsilon_h^{(\nu-1)}\right\|_t + K_2 \left\|e_h^{(\nu-1)}\right\|_t, \end{aligned} \tag{3.1}$$

$t \in I_a$, $\nu = 1, 2, \ldots$. Putting $\tilde{v}^{(\nu)}(t) = \|z^{(\nu)} - \tilde{z}^{(\nu)}\|_t$ and taking into account that

$$\left\|y^{(\nu)} - \tilde{y}^{(\nu)}\right\|_t \leq \int_a^t \|z^{(\nu)} - \tilde{z}^{(\nu)}\|_s \, ds$$

the inequality (3.1) can be rewritten in the form

$$\begin{aligned}\tilde{v}^{(\nu)}(t) \leq{}& A \int_a^t \tilde{v}^{(\nu)}(s)\, ds + B \int_a^t \tilde{v}^{(\nu-1)}(s)\, ds \\ &+ K\tilde{v}^{(\nu-1)}(t) + w_h^{(\nu-1)}(t),\end{aligned}$$

where A, B, and K are defined as in Section 1 and

$$w_h^{(\nu-1)}(t) = B \left\|\epsilon_h^{(\nu-1)}\right\|_t + K \left\|e_h^{(\nu-1)}\right\|_t.$$

Define $\tilde{u}^{(\nu)}(t)$ by

$$\begin{cases} \tilde{u}^{(\nu)}(t) = A\int_a^t \tilde{u}^{(\nu)}(s)\,ds + B\int_a^t \tilde{u}^{(\nu-1)}(s)\,ds \\ \qquad\qquad + K\tilde{u}^{(\nu-1)}(t) + w_h^{(\nu-1)}(t), \\ \tilde{u}^{(0)}(t) = \tilde{v}^{(0)}(t), \end{cases} \tag{3.2}$$

$t \in I_a$, $\nu = 1, 2, \ldots$. Then

$$\tilde{v}^{(\nu)}(t) \le \tilde{u}^{(\nu)}(t), \qquad t \in I_a, \qquad \nu = 1, 2, \ldots .$$

Proceeding similarly as in Section 1, it can be verified that the solution $\tilde{u}^{(\nu)}(t)$ to (3.2) has the form

$$\begin{aligned} \tilde{u}^{(\nu)}(t) = M\int_a^t e^{A(t-s)}\tilde{u}^{(\nu-1)}(s)\,ds \\ + K\tilde{u}^{(\nu-1)}(t) + A\int_a^t e^{A(t-s)} w_h^{(\nu-1)}(s)\,ds \\ + w_h^{(\nu-1)}(t), \end{aligned}$$

$t \in I_a$, $\nu = 1, 2, \ldots,$ where $M = B + AK$. Multiplying the above equation by $e^{-A(t-a)}$ we obtain

$$\begin{cases} \hat{u}^{(\nu)}(t) = M\int_a^t \hat{u}^{(\nu-1)}(s)\,ds + K\hat{u}^{(\nu-1)}(t) \\ \qquad\qquad + \eta_h^{(\nu-1)}(t), \\ \hat{u}^{(0)}(t) = e^{-A(t-a)}\tilde{u}^{(0)}(t), \end{cases} \tag{3.3}$$

$t \in I_a$, $\nu = 1, 2, \ldots,$ where

$$\hat{u}^{(\nu)}(t) = e^{-A(t-a)}\tilde{u}^{(\nu)}(t),$$

and

$$\eta_h^{(\nu-1)}(t) = A\int_a^t e^{-A(s-a)} w_h^{(\nu-1)}(s)\,ds + e^{-A(t-a)} w_h^{(\nu-1)}(t).$$

Equation (3.3) can be written in the form

$$\hat{u}^{(\nu)} = (P+Q)\hat{u}^{(\nu-1)} + \eta_h^{(\nu-1)}, \tag{3.4}$$

$\nu = 1, 2, \ldots,$ where P and Q are defined as before. The solution to (3.4) is

$$\hat{u}^{(\nu)} = (P+Q)^{\nu}\hat{u}^{(0)} + \sum_{i=0}^{\nu-1}(P+Q)^{\nu-i-1}\eta_h^{(i)},$$

and taking into account the form of the operators P and Q after simple calculations we obtain

$$\hat{u}^{(\nu)}(t) = \sum_{i=0}^{\nu} \binom{\nu}{i} M^i K^{\nu-i} \int_a^t \frac{(t-s)^{i-1}}{(i-1)!} \hat{u}^{(0)}(s)\, ds$$
$$+ \sum_{i=0}^{\nu-1} \sum_{j=0}^{i} \binom{i}{j} M^j K^{i-j} \int_a^t \frac{(t-s)^{j-1}}{(j-1)!} \eta_h^{(\nu-i-1)}(s)\, ds.$$

To obtain useful bounds on $\hat{u}^{(\nu)}(t)$ we further assume that

$$\tilde{u}^{(0)}(t) \le \tilde{Z}_0,$$
$$w_h^{(\nu-1)}(t) \le \Omega_h^{(\nu-1)},$$

$t \in I_a$, where $\tilde{Z}_0$ and $\Omega_h^{(\nu-1)}$ are independent of t. Then it is easy to verify that

$$\hat{u}^{(0)}(t) \le \tilde{Z}_0,$$
$$\eta_h^{(\nu-1)}(t) \le \Omega_h^{(\nu-1)},$$

$t \in I_a$, and it follows that

$$\hat{u}^{(\nu)}(t) \le \tilde{Z}_0 \sigma^{(\nu)}(t) + \sum_{i=0}^{\nu-1} \sigma^{(i)}(t) \Omega_h^{(\nu-i-1)}, \tag{3.5}$$

where $\sigma^{(\nu)}(t)$ is defined by (2.7).

We will summarize the above discussion in the following theorem.

THEOREM 3. *Assume that* $\|z^{(0)} - \tilde{z}^{(0)}\| \le \tilde{Z}_0$, $t \in I_a$. *Then*

$$\left\| y^{(\nu)} - \tilde{y}^{(\nu)} \right\|_t \le \left(\tilde{Y}_0 \sigma^{(\nu)}(t) + (b-a) \sum_{i=0}^{\nu-1} \sigma^{(i)}(t) \Omega_h^{(\nu-i-1)} \right) e^{A(t-a)}$$

and

$$\left\| z^{(\nu)} - \tilde{z}^{(\nu)} \right\|_t \le \left(\tilde{Z}_0 \sigma^{(\nu)}(t) + \sum_{i=0}^{\nu-1} \sigma^{(i)}(t) \Omega_h^{(\nu-i-1)} \right) e^{A(t-a)},$$

$\nu = 0, 1, \dots, t \in I_a$, *where* $\tilde{Y}_0 = (b-a)\tilde{Z}_0$, *and* $\sigma^{(\nu)}(t)$ *is defined by* (2.7).

In what follows we will need the following lemmas.

LEMMA 4. *Assume that* $0 \le K < 1$. *Then*

$$\sum_{s=0}^{\infty} \binom{j+s}{j} K^s = \frac{1}{(1-K)^{j+1}}, \qquad j = 0, 1, \dots . \tag{3.6}$$

Proof. The lemma is obviously true for $j = 0$. Assume it is true for j. Differentiating (3.6) with respect to K, we obtain

$$\sum_{s=1}^{\infty} s \binom{j+s}{j} K^{s-1} = \frac{j+1}{(1-K)^{j+2}}$$

or

$$\sum_{s=0}^{\infty} \frac{s+1}{j+1} \binom{j+1+s}{j} K^s = \frac{1}{(1-K)^{j+2}}.$$

Since

$$\frac{s+1}{j+1} \binom{j+1+s}{j} = \binom{j+1+s}{j+1}$$

the penultimate equation is equivalent to (3.6) with j replaced by $j+1$. ∎

LEMMA 5. *The following estimate holds*

$$\sum_{i=0}^{\nu-1} \sigma^{(i)}(t) \le \frac{1}{1-K} \sum_{j=0}^{\nu-1} \frac{1}{j!} \left(\frac{M(t-a)}{1-K} \right)^j,$$

$\nu = 1, 2, \dots, t \in I_a$.

Proof. Changing the order of summation and using Lemma 4, we obtain

$$\begin{aligned}
\sum_{i=0}^{\nu-1} \sigma^{(i)}(t) &= \sum_{i=0}^{\nu-1} \sum_{j=0}^{i} \binom{i}{j} \frac{(M(t-a))^j}{j!} K^{i-j} \\
&= \sum_{j=0}^{\nu-1} \sum_{i=j}^{\nu-1} \binom{i}{j} \frac{(M(t-a))^j}{j!} K^{i-j} \\
&= \sum_{j=0}^{\nu-1} \frac{(M(t-a))^j}{j!} \sum_{s=0}^{\nu-1-j} \binom{j+s}{j} K^s \\
&\le \sum_{j=0}^{\nu-1} \frac{(M(t-a))^j}{j!} \sum_{s=0}^{\infty} \binom{j+s}{j} K^s \\
&= \frac{1}{1-K} \sum_{j=0}^{\nu-1} \frac{1}{j!} \left(\frac{M(t-a)}{1-K} \right)^j. \quad ∎
\end{aligned}$$

Remark. It follows from Lemma 5 that the series $\sum_{i=0}^{\infty} \sigma^{(i)}(t)$ is convergent and it can be easily verified that

$$\sum_{i=0}^{\infty} \sigma^{(i)}(t) = \frac{1}{1-K} \exp\left(\frac{M(t-a)}{1-K} \right).$$

In particular, $\sigma^{(i)}(t) \to 0$ as $i \to \infty$ uniformly in $t \in I_a$.

To refine further the bounds given in Theorem 3 we will make additional assumptions on the sequence $\Omega_h^{(\nu)}$, $\nu = 0, 1, \dots$. We have the following theorems.

THEOREM 6. *Assume that* $\Omega_h^{(\nu)} \le \Omega_h$, $\nu = 0, 1, \dots,$ *where* Ω_h *is independent of* ν *and* $\Omega_h \to 0$ *as* $h \to 0$. *Assume also that* $\|z^{(0)} - \tilde{z}^{(0)}\|_t \le \tilde{Z}_0$, $t \in I_a$. *Then*

$$\left\| y^{(\nu)} - \tilde{y}^{(\nu)} \right\|_t \le \left(\tilde{Y}_0 \sigma^{(\nu)}(t) + \frac{\Omega_h (b-a)}{1-K} \sum_{i=0}^{\nu-1} \frac{1}{i!} \left(\frac{M(t-a)}{1-K} \right)^i \right) e^{A(t-a)},$$

and

$$\left\| z^{(\nu)} - \tilde{z}^{(\nu)} \right\|_t \le \left(\tilde{Z}_0 \sigma^{(\nu)}(t) + \frac{\Omega_h}{1-K} \sum_{i=0}^{\nu-1} \frac{1}{i!} \left(\frac{M(t-a)}{1-K} \right)^i \right) e^{A(t-a)},$$

$\nu = 0, 1, \dots,$ $t \in I_a$, *where* $\tilde{Y}_0 = (b-a)\tilde{Z}_0$, *and* $\sigma^{(\nu)}(t)$ *is defined by* (2.7). *In particular,* $\|y^{(\nu)} - \tilde{y}^{(\nu)}\|_t \to 0$ *and* $\|z^{(\nu)} - \tilde{z}^{(\nu)}\|_t \to 0$ *as* $\nu \to \infty$ *and* $h \to 0$.

THEOREM 7. *Assume that* $\Omega_h^{(\nu)} \le D\sigma^{(\nu)}(b)$, D *some constant,* $\nu = 0, 1, \dots,$ *and that* $\|z^{(0)} - \tilde{z}^{(0)}\|_t \le \tilde{Z}_0$, $t \in I_a$. *Then*

$$\left\| y^{(\nu)} - \tilde{y}^{(\nu)} \right\|_t \le \left(\tilde{Y}_0 \sigma^{(\nu)}(t) + (b-a) \sum_{i=0}^{\nu-1} \sigma^{(i)}(t) \sigma^{(\nu-i-1)}(b) \right) e^{A(t-a)},$$

and

$$\left\| z^{(\nu)} - \tilde{z}^{(\nu)} \right\|_t \le \left(\tilde{Z}_0 \sigma^{(\nu)}(t) + \sum_{i=0}^{\nu-1} \sigma^{(i)}(t) \sigma^{(\nu-i-1)}(b) \right) e^{A(t-a)},$$

$\nu = 0, 1, \dots, t \in I_a$. *In particular,* $\|y^{(\nu)} - \tilde{y}^{(\nu)}\|_t \to 0$ *and* $\|z^{(\nu)} - \tilde{z}^{(\nu)}\|_t \to 0$ *as* $\nu \to \infty$ *and* $h \to 0$.

Proof. Since $\sigma^{(\nu)}(t) > 0$ and the series $\sum_{i=0}^{\infty} \sigma^{(i)}(t)$ is convergent it follows from Cauchy's theorem on the product of convergent series that the series

$$\sum_{\nu=0}^{\infty} \sum_{i=0}^{\nu-1} \sigma^{(i)}(t)\,\sigma^{(\nu-i-1)}(b)$$

is also convergent. Hence,

$$\sum_{i=0}^{\nu-1} \sigma^{(i)}(t)\,\sigma^{(\nu-i-1)}(b) \to 0$$

as $\nu \to \infty$. ∎

In practical implementations the sequence $\Omega_h^{(\nu)}$ will usually satisfy a bound which is a combination of bounds examined in Theorems 6 and 7. Assume, for example, that we choose the stepsize and the order of numerical method used to compute $\tilde{y}_h^{(\nu)}$ and $\tilde{z}_h^{(\nu)}$ to keep the estimates of local discretization errors below the tolerance $\mathrm{TOL}^{(\nu)}$ which depends on the iteration index ν. If this tolerance has the form

$$\mathrm{TOL}^{(\nu)} = \max\bigl(\mathrm{TOL}, \sigma^{(\nu)}(b)\bigr), \tag{3.7}$$

where TOL is the tolerance related to the accuracy requested by the user, then it will be demonstrated in the next section that if the method is convergent then $\Omega_h^{(\nu)}$ satisfies the bound

$$\Omega_h^{(\nu)} \le D\,\mathrm{TOL}^{(\nu)}, \tag{3.8}$$

where D is a constant which depends only on the problem under consideration.

If $\mathrm{TOL}^{(\nu)}$ is defined by (3.7), then we can use a larger stepsize and save on the number of function evaluations when $\sigma^{(\nu)}(b) > \mathrm{TOL}$, which usually happens in the first few iterations. For sufficiently large ν we will have $\mathrm{TOL}^{(\nu)} = \mathrm{TOL}$ and this forces us to use more stringent stepsizes in the corresponding sweeps of waveform relaxation.

Let ν_1 be an iteration index such that

$$\mathrm{TOL}^{(\nu)} = \begin{cases} \sigma^{(\nu)}(b), & \nu < \nu_1, \\ \mathrm{TOL}, & \nu \ge \nu_1, \end{cases}$$

and assume (3.8). Then it follows that

$$\|y^{(\nu)} - \tilde{y}^{(\nu)}\|_t \le \left(\tilde{Y}_0 \sigma^{(\nu)}(t) + \frac{D(b-a)\mathrm{TOL}}{1-K} \sum_{j=0}^{\nu-\nu_1-1} \frac{1}{j!} \left(\frac{M(t-a)}{1-K} \right)^j + D(b-a) \sum_{j=0}^{\nu_1-1} \sigma^{(j)}(b) \sigma^{(\nu-j-1)}(t) \right) e^{A(t-a)},$$

and

$$\|z^{(\nu)} - \tilde{z}^{(\nu)}\|_t \le \left(\tilde{z}_0 \sigma^{(\nu)}(t) + \frac{D\,\mathrm{TOL}}{1-K} \sum_{j=0}^{\nu-\nu_1-1} \frac{1}{j!} \left(\frac{M(t-a)}{1-K} \right)^j + D \sum_{j=0}^{\nu_1-1} \sigma^{(j)}(b) \sigma^{(\nu-j-1)}(t) \right) e^{A(t-a)},$$

$\nu \ge \nu_1$, $t \in I_a$, and we can conclude that $\|y^{(\nu)} - \tilde{y}^{(\nu)}\|_t \to 0$ and $\|z^{(\nu)} - \tilde{z}^{(\nu)}\|_t \to 0$ as $\mathrm{TOL} \to 0$.

Combining Theorem 1 with the results of this section we can also conclude that $\tilde{y}^{(\nu)} \to y$ and $\tilde{z}^{(\nu)} \to y'$ as $\nu \to \infty$ and $h \to 0$ if $\Omega_h^{(\nu)}$ satisfies the bounds given in Theorem 6 and 7, or as $\mathrm{TOL} \to 0$ if $\Omega_h^{(\nu)}$ satisfies (3.8) with $\mathrm{TOL}^{(\nu)}$ given by (3.7).

4. CONVERGENCE OF NUMERICAL WR ITERATIONS $\tilde{y}_h^{(\nu)}$ AND $\tilde{z}_h^{(\nu)}$

In this section we will derive the bounds for $\|\tilde{y}^{(\nu)} - \tilde{y}_h^{(\nu)}\|_t$ and $\|\tilde{z}^{(\nu)} - \tilde{z}_h^{(\nu)}\|_t$ where $\tilde{y}_h^{(\nu)}$ and $\tilde{z}_h^{(\nu)}$ are numerical approximations to $\tilde{y}^{(\nu)}$ and $\tilde{z}^{(\nu)}$, respectively, obtained by application to (1.3) of quasilinear multistep methods for functional differential equations of neutral type [10]. These methods are defined by

$$\begin{cases} \tilde{y}_h^{(\nu)}(t_{i+k-1} + rh) + \sum_{j=0}^{k-1} a_j(r) \tilde{y}_h^{(\nu)}(t_{i+j}) \\ \qquad = h\phi\big(t_i, \tilde{y}_h^{(\nu)}(\cdot), \tilde{y}_h^{(\nu-1)}(\cdot), \tilde{z}_h^{(\nu)}(\cdot), \tilde{z}_h^{(\nu-1)}(\cdot), r, h\big), \\ \tilde{z}_h^{(\nu)}(t_{i+k-1} + rh) = \psi\big(t_i, \tilde{y}_h^{(\nu)}(\cdot), \tilde{y}_h^{(\nu-1)}(\cdot), \tilde{z}_h^{(\nu)}(\cdot), \tilde{z}_h^{(\nu-1)}(\cdot), r, h\big), \end{cases} \tag{4.1}$$

$i = 0, 1, \ldots, N - k, r \in (0, 1]$, and it is assumed that $\tilde{y}_h^{(\nu)}(t)$ and $\tilde{z}_h^{(\nu)}(t)$ are given for $t \in [\alpha, t_{k-1}]$. Here $t_i = a + ih$, $i = 0, 1, \ldots, N$, $Nh = b - a$, the functions $a_j(r)$, $j = 0, 1, \ldots, k - 1$, are continuous, $a_{k-1}(0) = -1$, $a_j(0) = 0$, $j = 0, 1, \ldots, k - 2$, and the increment functions ϕ and ψ satisfy the following conditions.

(a)

$$\phi(t, y(\cdot), y(\cdot), z(\cdot), z(\cdot), 0, 0) = 0$$

and

$$\begin{aligned}\|\phi(t, y_1(\cdot), y_2(\cdot), z_1(\cdot), z_2(\cdot), r, h) - \phi(t, \bar{y}_1(\cdot), \bar{y}_2(\cdot), \bar{z}_1(\cdot), \bar{z}_2(\cdot), r, h)\| \\ \leq M_1(\|y_1 - \bar{y}_1\|_{t+kh} + \|z_1 - \bar{z}_1\|_{t+kh}) \\ + M_2(\|y_2 - \bar{y}_2\|_{t+kh} + \|z_2 - \bar{z}_2\|_{t+kh})\end{aligned}$$

with $M_1, M_2 \geq 0$ for

$$t \in [a, b - kh], y_1, \bar{y}_1, y_2, \bar{y}_2 \in C_g[I_\alpha, R^n], z_1, \bar{z}_1, z_2, \bar{z}_2 \in C_{g'}[I_\alpha, R^n].$$

(b)

$$\begin{aligned}\|\psi(t, y_1(\cdot), y_2(\cdot), z_1(\cdot), z_2(\cdot), r, h) - \psi(t, \bar{y}_1(\cdot), \bar{y}_2(\cdot), \bar{z}_1(\cdot), \bar{z}_2(\cdot), r, h)\| \\ \leq L_1\|y_1 - \bar{y}_1\|_{t+kh} + L_2\|y_2 - \bar{y}_2\|_{t+kh} \\ + K_1\|z_1 - \bar{z}_2\|_{t+kh} + K_2\|z_2 - \bar{z}_2\|_{t+kh}, \\ L_1, L_2 \geq 0, 0 \leq K_1 + K_2 < 1, \\ t \in [a, b - kh]; y_1, \bar{y}_1, y_2, \bar{y}_2 \in C_g[I_\alpha, R^n]; z_1, \bar{z}_1, z_2, \bar{z}_2 \in C_{g'}[I_\alpha, R^n].\end{aligned}$$

We can assume without loss of generality that the constants L_1, L_2, K_1, and K_2 are the same as those appearing in Lipschitz condition for the function F.

The class of quasilinear multistep methods (4.1) is quite general and includes as special cases most discrete variable methods for neutral functional differential equations. In particular, it includes Runge–Kutta, linear multistep, and predictor-corrector methods for (1.1), see [10].

We define the local discretization errors $\eta_h^{(\nu)}(t_i, r)$ and $\xi_h^{(\nu)}(t_i, r)$ of (4.1) as residua obtained by replacing $\tilde{y}_h^{(\nu)}$ and $\tilde{z}_h^{(\nu)}$ by $\tilde{y}^{(\nu)}$ and $\tilde{z}^{(\nu)}$, i.e.,

$$\begin{cases} \tilde{y}^{(\nu)}(t_{i+k-1} + rh) + \sum_{j=0}^{k-1} a_j(r)\tilde{y}^{(\nu)}(t_{i+j}) \\ \qquad = h\phi\big(t_i, \tilde{y}^{(\nu)}(\cdot), \tilde{y}_h^{(\nu-1)}(\cdot), \tilde{z}^{(\nu)}(\cdot), \tilde{z}_h^{(\nu-1)}(\cdot), r, h\big) + \eta_h^{(\nu)}(t_i, r), \\ \tilde{z}^{(\nu)}(t_{i+k-1} + rh) = \psi\big(t_i, \tilde{y}^{(\nu)}(\cdot), \tilde{y}_h^{(\nu-1)}(\cdot), \tilde{z}^{(\nu)}(\cdot), \tilde{z}_h^{(\nu-1)}(\cdot), r, h\big) \\ \qquad + \xi_h^{(\nu)}(t_i, r), \end{cases} \tag{4.2}$$

$i = 0, 1, \dots, N-k$, $r \in (0, 1]$. We recall that $\tilde{y}_h^{(\nu-1)}$ and $\tilde{z}_h^{(\nu-1)}$ are approximations to $\tilde{y}^{(\nu-1)}$ and $\tilde{z}^{(\nu-1)}$ which were computed in the previous iteration sweep. We also define

$$\alpha_h^{(\nu)} = \sup\big\{\|\eta_h^{(\nu)}(t_i, r)\| : i = k-1, k, \dots, N-1, r \in (0,1]\big\},$$

$$\beta_h^{(\nu)} = \sup\big\{\|\eta_h^{(\eta)}(t_i, 1)\| : i = k-1, k, \dots, N-1\big\},$$

$$\gamma_h^{(\nu)} = \sup\big\{\|\xi_h^{(\nu)}(t_i, r)\| : i = k-1, k, \dots, N-1, r \in (0,1]\big\}.$$

The method (4.1) is consistent (with (1.3) on the solution $\tilde{y}^{(\nu)}$) if $\alpha_h^{(\nu)} = o(1)$, $\beta_h^{(\nu)} = o(h)$, and $\gamma_h^{(\nu)} = o(1)$ as $h \to 0$. The method (4.1) has order p if $\alpha_h^{(\nu)} = O(h^p)$, $\beta_h^{(\nu)} = O(h^{p+1})$, and $\gamma_h^{(\nu)} = O(h^p)$ as $h \to 0$. It has strong order p if $\alpha_h^{(\nu)} = O(h^{p+1})$, $\beta_h^{(\nu)} = O(h^{p+1})$, and $\gamma_h^{(\nu)} = O(h^{p+1})$ as $h \to 0$. The method (4.1) is stable if no root of the polynomial

$$\rho(\theta) = \theta^k + \sum_{j=0}^{k-1} a_j(1)\theta^j$$

has modulus greater than one and every root with modulus one is simple.

Consider the nonhomogeneous recurrence equation

$$Z_{i+k} + \sum_{j=0}^{k-1} a_j(1) Z_{i+j} = u_i, \tag{4.3}$$

$i = 0, 1, \dots$. We will need the following lemma, the proof of which can be found in [23].

LEMMA 8. *Assume that the method* (4.1) *is stable. Then there exists a constant* $C \geq 1$ *such that every solution of* (4.3) *satisfies the inequality*

$$\max_{0 \leq s \leq k-1} \|Z_{i+s}\| \leq C\left(\max_{0 \leq s \leq k-1} \|Z_s\| + \sum_{s=0}^{i-1} \|u_s\|\right),$$

$i = 0, 1, \dots$.

We are now ready to examine the bounds on the global discretization errors $\|\tilde{y}^{(\nu)} - \tilde{y}_h^{(\nu)}\|_t$ and $\|\tilde{z}^{(\nu)} - \tilde{z}_h^{(\nu)}\|_t$ of the method (4.1). Although these bounds can be obtained by standard techniques such as those described in [7, 9, 10], we will briefly sketch their derivation for the sake of completeness. To simplify the presentation we will assume constant stepsize implementation and we refer to [21] for the convergence discussion of variable step variable order quasilinear multistep methods for neutral equations (1.1).

Subtracting (4.1) from (4.2) for $r = 1$ and using Lemma 8 and the Lipschitz condition for the increment function ϕ, we get

$$\max_{0 \le s \le k-1} \left\| \epsilon_h^{(\nu)}(t_{i+s}) \right\| \le C \Bigg(\max_{0 \le s \le k-1} \left\| \epsilon_h^{(\nu)}(t_s) \right\| + hM_1 \sum_{s=0}^{i-1} \left(\left\| \epsilon_h^{(\nu)} \right\|_{t_{s+k}} + \left\| e_h^{(\nu)} \right\|_{t_{s+k}} \right) + \sum_{s=0}^{i-1} \left\| \eta_h^{(\nu)}(t_s, 1) \right\| \Bigg), \quad (4.4)$$

$i = 0, 1, \ldots, N-k$. Subtracting (4.1) from (4.2) for $r \in (0, 1]$ and using the fact that $\|\epsilon_h^{(\nu)}\|_t$ and $\|{}_h^{(\nu)}\|_t$ are nondecreasing with respect to t, we have also

$$\left\| \epsilon_h^{(\nu)} \right\|_{t_{i+k}} \le A^* \left\| \epsilon_h^{(\nu)} \right\|_{t_{i+k-1}} + hM_1 \left(\left\| \epsilon_h^{(\nu)} \right\|_{t_{i+k}} + \left\| e_h^{(\nu)} \right\|_{t_{i+k}} \right) + \alpha_h^{(\nu)}, \quad (4.5)$$

and

$$\left\| \epsilon_h^{(\nu)} \right\|_{t_{i+k}} \le A^* C \Bigg(\left\| \epsilon_h^{(\nu)} \right\|_{t_{k-1}} + hM_1 \sum_{s=0}^{i} \left(\left\| \epsilon_h^{(\nu)} \right\|_{t_{s+k}} + \left\| e_h^{(\nu)} \right\|_{t_{s+k}} \right) + \alpha_h^{(\nu)} + (t_i - a) \beta_h^{(\nu)}(h) \Bigg), \quad (4.6)$$

$i = 0, 1, \ldots, N-k$, where

$$A^* = \sup \left\{ \sum_{j=0}^{k-1} \left| a_j(r) \right| : r \in (0, 1] \right\}.$$

Using the Lipschitz condition for the increment function ψ we have also

$$\left\| e_h^{(\nu)} \right\|_{t_{s+k}} \le \left\| e_h^{(\nu)} \right\|_{t_{k-1}} + L_1 \left\| \epsilon_h^{(\nu)} \right\|_{t_{s+k}} + K_1 \left\| e_h^{(\nu)} \right\|_{t_{s+k}} + \gamma_h^{(\nu)},$$

$s = 0, 1, \ldots, N-k$. Hence,

$$\left\| e_h^{(\nu)} \right\|_{t_{s+k}} \le \frac{1}{1 - K_1} \left\| e_h^{(\nu)} \right\|_{t_{k-1}} + \frac{L_1}{1 - K_1} \left\| \epsilon_h^{(\nu)} \right\|_{t_{s+k}} + \frac{1}{1 - K_1} \gamma_h^{(\nu)}, \quad (4.7)$$

and subtracting (4.7) from (4.6), it follows that

$$\begin{aligned}\left\|\epsilon_h^{(\nu)}\right\|_{t_{i+k}} \leq A^*C\Bigg(&\left\|\epsilon_h^{(\nu)}\right\|_{t_{k-1}} + \frac{M_1(t_i-a)}{1-K_1}\left\|e_h^{(\nu)}\right\|_{t_{k-1}}\\ &+h\,\frac{M_1(1+L_1-K_1)}{1-K_1}\sum_{s=0}^{i}\left\|\epsilon_h^{(\nu)}\right\|_{t_{s+k}} + \alpha_h^{(\nu)}\\ &+\frac{(t_i-a)\,\beta_h^{(\nu)}}{h} + \frac{M_1(t_i-a)}{1-K_1}\,\gamma_h^{(\nu)}\Bigg),\end{aligned}$$

$i = 0, 1, \ldots, N-k$. Let

$$D = A^*C\max\left\{1,\ \frac{M_1(b-a)}{1-K_1},\quad \frac{M_1(1+L_1-K_1)}{1-K_1}\right\}.$$

Then

$$\begin{aligned}\left\|\epsilon_h^{(\nu)}\right\|_{t_{i+k}} \leq D\Bigg(&\left\|\epsilon_h^{(\nu)}\right\|_{t_{k-1}} + \left\|e_h^{(\nu)}\right\|_{t_{k-1}}\\ &+h\sum_{s=0}^{i}\left\|\epsilon_h^{(\nu)}\right\|_{t_{s+k}} + \alpha_h^{(\nu)} + \beta_h^{(\nu)}/h + \gamma_h^{(\nu)}\Bigg),\end{aligned}$$

$i = 0, 1, \ldots, N-k$. Let h_0 be such that $1 - h_0 D > 0$. Then for $h < h_0$ we can eliminate $\|\epsilon_h^{(\nu)}\|_{t_{i+k}}$ from the right hand side of the above inequality to get

$$\begin{aligned}\left\|\epsilon_h^{(\nu)}\right\|_{t_{t+k}} \leq D^*\Bigg(&\left\|\epsilon_h^{(\nu)}\right\|_{t_{k-1}} + \left\|e_h^{(\nu)}\right\|_{t_{k-1}}\\ &+h\sum_{s=0}^{i-1}\left\|\epsilon_h^{(\nu)}\right\|_{t_{s+k}} + \alpha_h^{(\nu)} + \beta_h^{(\nu)}/h + \gamma_h^{(\nu)}\Bigg),\end{aligned}$$

$i = 0, 1, \ldots, N-k$, $h < h_0$, where $D^* = D/(1 - h_0 D)$. Let Z_{i+k} be the sequence defined by the equation

$$\begin{aligned}Z_{i+k} = D^*\Bigg(&\left\|\epsilon_h^{(\nu)}\right\|_{t_{k-1}} + \left\|e_h^{(\nu)}\right\|_{t_{k-1}} + h\sum_{s=0}^{i-1} Z_{s+k}\\ &+\alpha_h^{(\nu)} + \beta_h^{(\nu)}/h + \gamma_h^{(\nu)}\Bigg),\end{aligned} \tag{4.8}$$

$i = 0, 1, \dots, N-k$. Then $\|\epsilon_h^{(\nu)}\|_{t_{i+k}} \le Z_{i+k}$. The solution of (4.8) is given

$$Z_{i+k} = (1 + hD^*)^i Z_k,$$

$i = 0, 1, \dots, N-k$. Since $(1 + hD^*)^i \le \exp(D^*(b-a))$ we obtain

$$\begin{aligned}\|\epsilon_h^{(\nu)}\|_b \le D^*\Big(&\|\epsilon_h^{(\nu)}\|_{t_{k-1}} + \|e_h^{(\nu)}\|_{t_{k-1}} + \alpha_h^{(\nu)} + \beta_h^{(\nu)}/h + \gamma_h^{(\nu)}\Big)\\ &\times \exp(D^*(b-a)),\end{aligned}$$

$i = 0, 1, \dots, N-k$. Using (4.7) we have also

$$\begin{aligned}\|e_h^{(\nu)}\|_b \le E^*\Big(&\|\epsilon_h^{(\nu)}\|_{t_{k-1}} + \|e_h^{(\nu)}\|_{t_{k-1}} + \alpha_h^{(\nu)} + \beta_h^{(\nu)}/h + \gamma_h^{(\nu)}\Big)\\ &\times \exp(D^*(b-a)),\end{aligned}$$

where

$$E^* = \frac{L_1 D^* + 1}{1 - K_1}.$$

Similarly, we can obtain estimates for $\|\epsilon_h^{(\nu)}\|_t$ and $\|e_h^{(\nu)}\|_t$ where t is any point from the interval $[t_{k-1}, b]$. We summarize the above discussion in the following theorem.

THEOREM 9. *Assume that the method* (4.1) *is consistent and stable. Then the global errors* $\|\epsilon_h^{(\nu)}\|_t$ *and* $\|e_h^{(\nu)}\|_t$ *can be bounded by*

$$\begin{aligned}\|\epsilon_h^{(\nu)}\|_t \le D^*\Big(&\|\epsilon_h^{(\nu)}\|_{t_{k-1}} + \|e_h^{(\nu)}\|_{t_{k-1}} + \alpha_h^{(\nu)} + \beta_h^{(\nu)}/h + \gamma_h^{(\nu)}\Big)\\ &\times \exp(D^*(t-a)),\end{aligned}$$

and

$$\begin{aligned}\|e_h^{(\nu)}\|_t \le E^*\Big(&\|\epsilon_h^{(\nu)}\|_{t_{k-1}} + \|e_h^{(\nu)}\|_{t_{k-1}} + \alpha_h^{(\nu)} + \beta_h^{(\nu)}/h + \gamma_h^{(\nu)}\Big)\\ &\times \exp(D^*(t-a)),\end{aligned}$$

$t \in [t_{k-1}, b]$, $\nu = 1, 2, \dots,$ *where the constants* D^* *and* E^* *are independent of* ν. *In particular, if* $\|\epsilon_h^{(\nu)}\|_{t_{k-1}} \to 0$ *and* $\|e_h^{(\nu)}\|_{t_{k-1}} \to 0$ *as* $h \to 0$, *then* $\tilde{y}_h^{(\nu)} - \tilde{y}^{(\nu)} \to 0$ *and* $\tilde{z}_h^{(\nu)} - \tilde{z}^{(\nu)} \to 0$ *as* $h \to 0$.

As observed before the bounds for global errors $\|y - \tilde{y}_h^{(\nu)}\|_t$ and $\|y' - \tilde{z}_h^{(\nu)}\|_t$ can be obtained using the triangle inequality to yield

$$\|y - \tilde{y}_h^{(\nu)}\|_t \le \|y - y^{(\nu)}\|_t + \|y^{(\nu)} - \tilde{y}^{(\nu)}\|_t + \|\tilde{y}^{(\nu)} - \tilde{y}_h^{(\nu)}\|_t$$

and

$$\|y' - \tilde{z}_h^{(\nu)}\|_t \le \|y' - z^{(\nu)}\|_t + \|z^{(\nu)} - \tilde{z}^{(\nu)}\|_t + \|\tilde{z}^{(\nu)} - \tilde{z}_h^{(\nu)}\|_t,$$

where the bounds on the corresponding terms appearing on the right hand sides of these inequalities are obtained in Theorems 1, 6, or 7, and 8. We can also conclude that $\tilde{y}_h^{(\nu)} \to y$ and $\tilde{z}_h^{(\nu)} \to y'$ as $\nu \to \infty$ and $h \to 0$ if the sequence $\Omega_h^{(\nu)}$ satisfies the conditions given in Theorem 6 or 7, or as TOL $\to 0$ if $\Omega_h^{(\nu)}$ satisfies (3.8) with $\mathrm{TOL}^{(\nu)}$ given by (3.7).

5. IMPLEMENTATION ASPECTS

In this section we describe the algorithm for the solution of (1.1) by the waveform relaxation technique. We will describe this algorithm for systems of delay differential equations of neutral type

$$\begin{cases} y_i'(t) = f_i(t, y(t), y(\alpha(t)), y'(\beta(t))), & t \in I_a, \\ y_i(t) = g_i(t), & t \in [\alpha, a], \end{cases} \tag{5.1}$$

$\alpha \le a$, $i = 1, 2, \dots, n$, where $g_i(t)$ are given initial functions and

$$y(t) = [y_1(t), \dots, y_n(t)]^T,$$

$$y(\alpha(t)) = [y_1(\alpha_1(t)), \dots, y_n(\alpha_n(t))]^T,$$

$$y'(\beta(t)) = [y_1'(\beta_1(t)), \dots, y_n'(\beta_n(t))]^T,$$

with $\alpha \le \alpha_i(t) \le t$, $\alpha \le \beta_i(t) \le t$, $i = 1, 2, \dots, n, t \in I_a$. Observe that each component y_i of y and y_i' of y' can depend only one one delay $\alpha_i(t)$ or $\beta_i(t)$. This is not the most general formulation for systems of delay equations, but by the process of augmenting described in [29] or [30], most systems can be reduced to this form at the expense of increasing the dimension n.

Consider the continuous-time Gauss−Jacobi iterations

$$z_i^{(\nu)}(t) = F_i\big(t, y^{(\nu)}(t), y^{(\nu-1)}(t), y^{(\nu)}(\alpha(t)), y^{(\nu-1)}(\alpha(t)), \\ z^{(\nu)}(\beta(t)), z^{(\nu-1)}(\beta(t))\big), \tag{5.2}$$

$\nu = 1, 2, \dots,$ where $z_i^{(\nu)}(t) = (d/dt)y_i^{(\nu)}(t)$, and $y_i^{(0)}(t)$ and $z_i^{(0)}(t)$ are given. They correspond to the splitting functions $F_i(t, y, \tilde{y}, u, \tilde{u}, v, \tilde{v})$ defined by

$$F_i\big(t, y, \tilde{y}, u, \tilde{u}, v, \tilde{v}\big) = f_i\Big(t, \tilde{y}_1, \dots, \tilde{y}_{i-1}, y_i, \tilde{y}_{i+1}, \dots, \tilde{y}_n, \\ \tilde{u}_1, \dots, \tilde{u}_{i-1}, u_i, \tilde{u}_{i+1}, \dots, \tilde{u}_n, \tilde{v}_1, \dots, \tilde{v}_{i-1}, v_i, \tilde{v}_{i+1}, \dots, \tilde{v}_n\Big),$$

$i = 1, 2, \ldots, n$. Consider also the perturbed iterations

$$\tilde{z}_i^{(\nu)}(t) = F_i\big(t, \tilde{y}^{(\nu)}(t), \tilde{y}_h^{(\nu-1)}(t), \tilde{y}^{(\nu)}(\alpha(t)), \tilde{y}_h^{(\nu-1)}(\alpha(t)), \tilde{z}^{(\nu)}(\beta(t)), \tilde{z}_h^{(\nu-1)}(\beta(t))\big), \quad (5.3)$$

$\nu = 1, 2, \ldots, \tilde{z}_h^{(\nu)}(t) = (d/dt)\tilde{y}_h^{(\nu)}(t)$, where $\tilde{z}_h^{(0)}$ is an approximation to $\tilde{z}^{(0)} = z^{(0)}$ and $\tilde{y}_h^{(0)}$ is an approximation to $\tilde{y}^{(0)} = y^{(0)}$. Assume that the approximations $\tilde{y}_{i,h}^{(\nu-1)}$ and $\tilde{z}_{i,h}^{(\nu-1)}$ to $\tilde{y}_i^{(\nu-1)}$ and $\tilde{z}_i^{(\nu-1)}$ are already computed and stored on the grid

$$a = t_{i,0}^{(\nu-1)} < t_{i,1}^{(\nu-1)} < \cdots < t_{i,N_i^{(\nu-1)}}^{(\nu-1)} = b$$

and that sufficiently accurate interpolation formulas are given to compute $\tilde{y}_{i,h}^{(\nu-1)}(t)$ and $\tilde{z}_{i,h}^{(\nu-1)}(t)$ between the grid points. We will describe how to compute the approximations $\tilde{y}_{i,h}^{(\nu)}$ and $\tilde{z}_{i,h}^{(\nu)}$ to the next iterations $\tilde{y}_i^{(\nu)}$ and $\tilde{z}_i^{(\nu)}$. We will use for this purpose the code SNDDELM for the numerical solution of neutral delay differential equations which was developed by Lo and Jackiewicz [22] and described in [21] and in the appendix to a recent book by Kuang [17]. This is a variable step/variable order algorithm based on the variable step formulation of Adams–Bashforth Adams–Moulton formulas of strong order p for (1.1), where p ranges from 1 to 12. The computed approximations $\tilde{y}_h^{(\nu)}$ and $\tilde{z}_h^{(\nu)}$ to $\tilde{y}^{(\nu)}$ and $\tilde{z}^{(\nu)}$ are stored in divided difference form. This representation allows for efficient and stable computation of the coefficients of the underlying Adams formulas from simple recursions. The method is implemented in PEICEI mode with local extrapolation, where the predictor step P uses the Adams–Bashforth method of strong order p and the corrector step C and interpolation step I uses the Adams–Moulton method of strong order p. This leads to the overall method of order p which, as explained in [10], possesses one term in the asymptotic expansions of global discretization errors. This allows us to obtain asymptotically correct estimates E_{p-2}, E_{p-1}, E_p, and E_{p+1} of local discretization errors corresponding to methods of orders $p-2$, $p-1$, p, and $p+1$. The estimates E_{p-2}, E_{p-1}, and E_p can be computed without any extra function evaluations using approximations already computed in the code. The computation of E_{p+1} involves one extra function evaluation. These estimates are used to control the stepsize and order of the algorithm. We refer to [21] for a complete description of this algorithm and underlying theoretical analysis.

Since, in general, the iterates $y^{(\nu)}$, $\tilde{y}^{(\nu)}$, $z^{(\nu)}$, and $\tilde{z}^{(\nu)}$ are for small indices ν quite far from y and y' it is not necessary in such a case to compute $\tilde{y}_h^{(\nu)}$ and $\tilde{z}_h^{(\nu)}$ to a high precision but only to an accuracy comparable to the errors $\|y - y^{(\nu)}\|_t$ and $\|z - z^{(\nu)}\|_t$. Since these errors can be bounded by a quantity proportional to $\sigma^{(\nu)}(t)$ (compare Theorem 1), where $\sigma^{(\nu)}(t)$ is defined by (2.7), it seems to be reasonable to choose a tolerance $\mathrm{TOL}^{(\nu)}$ defined by

$$\mathrm{TOL}^{(\nu)} = \max\bigl(\mathrm{TOL}, \sigma^{(\nu)}(b)\bigr) \tag{5.4}$$

to control the stepsize and the order of the algorithm while computing $\tilde{y}_h^{(\nu)}$ and $z_h^{(\nu)}$. Here TOL is the tolerance related to the desired accuracy. The tolerance $\mathrm{TOL}^{(\nu)}$ is quite large for small ν and becomes smaller as ν increases and the errors $\|y - y^{(\nu)}\|_t$ and $\|z - z^{(\nu)}\|_t$ decrease, and ultimately assumes the value TOL.

We iterate until there is no significant further change in the differences between the two successive iterations $\|\tilde{y}_h^{(\nu)} - \tilde{y}_h^{(\nu-1)}\|_t$ and $\|\tilde{z}_h^{(\nu)} - \tilde{z}_h^{(\nu-1)}\|_t$ and define $y_h = \tilde{y}_h^{(\nu)}$ and $z_h = \tilde{z}_h^{(\nu)}$. Alternatively, we could iterate until

$$\left\| \tilde{y}_h^{(\nu)} - \tilde{y}_h^{(\nu-1)} \right\|_t \le D^* \, \mathrm{TOL}^{(\nu)} \exp\bigl(D^*(b-a)\bigr)$$

and

$$\left\| \tilde{z}_h^{(\nu)} - \tilde{z}_h^{(\nu-1)} \right\|_t \le E^* \, \mathrm{TOL}^{(\nu)} \exp\bigl(D^*(b-a)\bigr),$$

where the constants D^* and E^* are defined in the proof of Theorem 8. However, these constants are difficult to determine a priori and the first strategy is much simpler and quite satisfactory in practice.

To compute $\tilde{y}_{i,h}^{(\nu)}$ and $\tilde{z}_{i,h}^{(\nu)}$, $i = 1, 2, \ldots, n$, we apply the code SNDDELM with the tolerance $\mathrm{TOL}^{(\nu)}$ defined by (5.4) to the system (5.3) in which all equations are independent of each other. Observe that the functions F_i appearing on the right hand side of (5.3) depend on $\tilde{y}_h^{(\nu-1)}(t)$, $\tilde{y}_h^{(\nu-1)}(\alpha(t))$, and $\tilde{z}_h^{(\nu-1)}(\beta(t))$ which were computed by the same code in the previous step corresponding to the iteration index $\nu - 1 > 0$. At the next step corresponding to the iteration index ν we generate the grids

$$a = t_{i,0}^{(\nu)} < t_{i,1}^{(\nu)} < \cdots < t_{i,N_i^{(\nu)}}^{(\nu)} = b$$

which, in general, differ from $t_{i,j}^{(\nu-1)}$, $i = 1, 2, \ldots, n$, $j = 0, 1, \ldots, N_i^{(\nu-1)}$. To compute the right hand sides of (5.3) it is necessary to provide approximations $\tilde{y}_{i,h}^{(\nu-1)}(t_{i,j}^{(\nu)})$, $\tilde{y}_{i,h}^{(\nu-1)}(\alpha_i(t_{i,j}^{(\nu)}))$ and $\tilde{z}_{i,h}^{(\nu-1)}(\beta_i(t_{i,j}^{(\nu)}))$, $i = 1, 2, \ldots, n$, $j = 0, 1, \ldots, N_i^{(\nu)}$, and, as mentioned above, this is done by interpolation of sufficiently high order. Such interpolation routines are integral parts of our code. We refer again to [21] for their complete description and analysis.

It has been observed in the context of ordinary differential equations that the ordering of equations strongly affects the number of iterations required to achieve convergence (compare [15, 33, 34]. Although there are some guidelines which often lead to good ordering, the determination of an optimal ordering for arbitrary systems is a very difficult problem and no attempt is made to address it in this paper.

The continuous-time Gauss–Jacobi iterations (5.2) decouple the system (5.1) into one-dimensional equations (blocks). In practice, when the dimension of the system is large, it may be advantageous to partition the system into subsystems of dimension larger than one and apply blockwise iteration. These blocks should be chosen in such a way that computational effort of each iteration is approximately the same for each subsystem. The experience gathered for systems of differential and algebraic-differential equations indicates that it is a good strategy to merge tightly coupled variables together. In practical application there may be a natural partitioning which corresponds to the physical characteristics of the system (compare [34] or [35]). We refer to Peterson and Mattison [31] for a survey of various partitioning strategies in transient analysis circuit simulation.

Another important aspect of waveform relaxation is windowing, i.e., dividing the interval of integration into smaller intervals called windows, on which the subsystems of (5.1) are solved in sequence. Although some dynamic windowing strategies have been proposed in the literature on waveform relaxation methods for differential systems (see [34, 36]) it is probably fair to say that this process is not fully understood and more work is needed in this area. In the numerical experiments presented in the next section we will not attempt to determine the optimal size of the window but divide the interval of integration into 1, 2, 4, or 8 subintervals and monitor how this affects the cost of the computational process.

6. NUMERICAL EXPERIMENTS

We have implemented the waveform relaxation algorithm described in the previous section on a sequential computer and tested it on many systems of delay differential equations listed in [14]. We define speedup_1 and speedup_2 as the ratios

$$\text{speedup}_1 = \frac{F_s}{F_p^1}, \qquad \text{speedup}_2 = \frac{F_s}{F_p^2},$$

where F_s is the number of function evaluations in a sequential implementation of SNDDELM algorithm, F_p^1 is the number of function evaluations of the most expensive component of the system, and F_p^2 is the total number of function evaluations divided by the number of components. These are, of course, theoretical speedups that would arise if each component was handled by a different processor and communication time was not taken into account. Moreover, we assume that evaluation time of each component is approximately the same. We present below the numerical results for the following two systems.

EXAMPLE 1 (Neves [29]).

$$\begin{aligned} y_1'(t) &= y_5(t-1) + y_3(t-1), \\ y_2'(t) &= y_1(t-1) + y_2(t-0.5), \\ y_3'(t) &= y_3(t-1) + y_1(t-0.5), \\ y_4'(t) &= y_5(t-1)\, y_4(t-1), \\ y_5'(t) &= y_1(t-1), \end{aligned}$$

$t \in [0, 1]$. Initial conditions:

$$\begin{aligned} y_1(t) &= y_4(t) = y_5(t) = \exp(t+1), \\ y_2(t) &= \exp(t+0.5), \\ y_3(t) &= \sin(t+1), \end{aligned}$$

$t \in [-1, 0]$.

Exact solution:

$$y_1(t) = \exp(t) - \cos(t) + e, \qquad t \in [0, 1],$$

$$y_2(t) = \begin{cases} 2\exp(t) + \exp(0.5) - 2, & t \in [0, 0.5], \\ \exp(t) + 2\exp(t-0.5) + t\exp(0.5) & \\ \qquad -2t + 1.5\exp(0.5) - 3, & t \in [0.5, 1], \end{cases}$$

$$y_3(t) = \begin{cases} \exp(t+0.5) - \cos(t) + 1 & \\ \qquad -\exp(0.5) + \sin(1), & t \in [0, 0.5], \\ -\cos(t) + \exp(t-0.5) - \sin(t-0.5) & \\ \qquad +(t+0.5)e - \exp(0.5) + \sin(1), & t \in [0.5, 1], \end{cases}$$

$$y_4(t) = 0.5\exp(2t) - 0.5 + e, \qquad t \in [0, 1],$$

$$y_5(5) = \exp(t) + e - 1, \qquad t \in [0, 1].$$

EXAMPLE 2 (Banks and Kappel [1]).

$$\begin{aligned} y_1'(t) &= 2y_2(t), \\ y_2'(t) &= -y_3(t) + y_1(t-1), \\ y_3'(t) &= 2y_2(t-1), \end{aligned}$$

$t \in [0, 3]$. Initial conditions:

$$y_1(t) = y_2(t) = y_3(t) = \begin{cases} 0, & t \in [-1, 0), \\ 1, & t = 0. \end{cases}$$

Exact solution:

$$y_1(t) = \begin{cases} 1 + 2t - t^2, & t \in [0, 1], \\ 2, & t \in [1, 3], \end{cases}$$

$$y_2(t) = \begin{cases} 1 - t, & t \in [0, 1], \\ 0, & t \in [1, 3], \end{cases}$$

$$y_3(t) = \begin{cases} 1, & t \in [0, 1], \\ -2 + 4t + t^2, & t \in [1, 2], \\ 2, & t \in [2, 3]. \end{cases}$$

Both examples (as well as the others not presented here) were solved for four different tolerances TOL $= 10^{-3}$, 10^{-6}, 10^{-9}, and 10^{-12} and for the number of windows $w = 1, 2, 4$, or 8. All windows were chosen of equal size; we did not attempt to choose their sizes adaptively since the techniques for doing this seem not to be well understood at this time, even for systems of ordinary differential equations. We present in Tables I and II the absolute error AERR of the waveform relaxation algorithm at the end of the interval of integration and theoretical speedup$_1$ and speedup$_2$.

The results presented in Table I are quite promising and indicate that waveform relaxation algorithms may be faster in this case when implemented on a parallel computer. On the other hand, the speedups presented in Table II are quite modest and probably the waveform relaxation algorithm would be, in fact, slower if communication time were taken into account. On some other examples, which are not presented here, the results were even less promising than those in Table II. We observed, however, they if we play a careful tolerance game trying to choose optimal tolerance for each component of the problem, and if we choose carefully the optimal number and sizes of windows, these theoretical speedups can be improved considerably. We observed that more stringent iteration tolerance should be used for loosely coupled problems. We also observed that more windows should be utilized if the number of steps needed for integration of smooth problems is large, and we should use a smaller number of windows for problems with derivative discontinuities which require a lot of step adjustment to restart the integration. We do not have at this time the general purpose algorithms for choosing the optimal

TABLE I
Example 1

TOL	w	AERR	Speedup$_1$	Speedup$_2$
	1	2.5E-2	3.05	5.67
10^{-3}	2	2.9E-2	2.53	5.02
	4	2.6E-4	1.83	3.38
	8	8.6E-3	1.81	2.32
	1	1.0E-6	2.47	5.15
10^{-6}	2	1.1E-7	3.07	5.66
	4	2.4E-4	2.68	5.07
	8	1.0E-7	2.23	3.87
	1	6.3E-9	2.73	4.10
10^{-9}	2	1.2E-9	3.42	5.46
	4	5.3E-10	2.13	5.07
	8	7.7E-11	2.46	4.66
	1	6.7E-12	2.41	4.28
10^{-12}	2	3.2E-11	3.02	4.69
	4	2.3E-10	3.79	6.45
	8	2.4E-12	2.02	5.60

TABLE II
Example 2

TOL	u	AERR	Speedup$_1$	Speedup$_2$
	1	2.5E-3	1.18	1.38
10^{-3}	2	2.3E-3	1.29	1.34
	4	4.26E-3	1.30	1.39
	8	7.0E-3	1.06	1.37
	1	1.9E-5	1.15	1.31
10^{-6}	2	6.7E-7	1.01	1.11
	4	1.2E-6	1.17	1.39
	8	7.5E-6	1.04	1.15
	1	2.9E-7	1.49	1.68
10^{-9}	2	6.3E-9	1.08	1.21
	4	2.9E-9	1.10	1.47
	8	4.3E-9	0.96	1.29
	1	4.5E-9	1.52	1.41
10^{-12}	2	2.8E-12	1.10	1.29
	4	5.4E-12	1.04	1.50
	8	4.1E-12	0.89	1.32

iteration tolerance and optimal number and sizes of windows and these topics will be the subject of future research. We also plan to investigate the acceleration of convergence of waveform relaxation iterations for delay differential systems. This was already examined in the context of ordinary differential equations by Nevanlinna [28] and in the context of parabolic partial differential equations by Vandewalle [33].

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